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ロバスト Wardrop 均衡問題と二次錐相補性問題への変換

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概要 与えられた交通ネットワークに対して，出発地と目的地のペア，および各々のペアに対する交通需要が与えられたとき，各ルートにおける交通量の均衡状態として知られるのが Wardrop 均衡である．Wardrop 均衡とは，いずれのドライバーも，自分だけがルートを変更することにより，所要時間を減少させることができない状態であり，このような均衡状態は Wardrop の等時間配分原理によって特徴づけられる．Wardrop 均衡の概念は，すべてのドライバーが交通ネットワークのすべての情報（所要時間関数のパラメータなど）をもち，それ自体がすべてのドライバーの共通認識であるという前提の下で意味をなす．しかし，現実においては，その前提が必ずしも満たされるとは限らない．そこで，本報告書では，各ドライバーが不確実な情報の下で起こり得る最悪のケースを想定し，それに基づいて自分のルートを選択するものと考える．そして，そのときに得られる均衡状態をロバスト Wardrop 均衡と定義する．本報告書では，ドライバーの目的地までの所要時間関数に不確実性を組み込み，その不確実性の下でロバスト Wardrop 均衡問題を定式化する．さらに，不確実性を表す集合が 2 ノルムで表されるとき，ロバスト Wardrop 均衡問題を二次錐相補性問題という既存の手法（平滑化ニュートン法など）で解くことのできるクラスの問題に再定式化する．最後に，具体的に与えられた交通ネットワークに対して，ロバスト Wardrop 均衡問題を計算機で実際に解き，不確実性集合の違いに対するロバスト Wardrop 均衡解の変化を調べる．

1 Introduction

Since the 1930s, the automobiles have been widely used all over the world because of the economic growth and the technological and scientific development. In order to make the automobile traffic more efficient, we need to design roadway infrastructures such as highways, traffic signals, and toll roads.

When we build new roads or decide new tolls on a traffic network, we need to forecast the traffic flow to estimate the effect due to such decisions. In general, all drivers are supposed to select the route with the minimum cost from the origin to the destination. In other words, the routes with positive traffic flow have the minimum cost, and more costly routes are not used. This flow distribution principle is called *Wardrop's user equilibrium principle* [20]. Also, the problem of finding a flow pattern satisfying Wardrop's user equilibrium principle is called the Traffic Assignment Problem (TAP). The TAP is formulated as mathematical programming problems such as a linear or nonlinear optimization problem, Variational Inequality Problem (VIP), Mixed Complementarity Problem (MCP), and Nonlinear Complementarity Problem (NCP) [1, 2, 4, 11, 17, 18].

In order to formulate the TAP as a mathematical programming problem, it is important to model the cost on each route appropriately. When the route cost function is expressed as the sum of road*[1] costs, the route cost function is called *additive* [1, 4, 17, 18]. Otherwise, it is called *non-additive* [2, 11].

In the TAP, we suppose that each user has complete information on the traffic network and can choose a route with minimum cost by using that information. However, in the real traffic network, each user's estimated cost can be often incorrect due to various uncertainties such as weather changeability or traffic accidents. Therefore he/she may choose a route with non-minimal cost, and the flow based on Wardrop's user equilibrium principle does not necessarily express the real network flow.

For the traffic model in which the drivers do not know the complete information on the network, the new concept called the *robust Wardrop equilibrium* [14, 15, 19] attracts much attention recently. In the robust Wardrop equilibrium, we assume that each driver can estimate the "*uncertainty set*" in which the uncertain data of his/her route cost function are contained, and then choose his/her route with taking the value of the *worst (route) cost function* into consideration. In other words, each driver

*[1]The road in a traffic network corresponds to the link in a directed graph. For more detail, see Section 2.

chooses his/her route based on the robust optimization policy [3, 6, 5, 13]. The traffic assignment problem based on the robust Wardrop equilibrium is called a robust TAP, which we will mainly discuss in the paper.

The robust Wardrop equilibrium has been studied by some researchers so far. Ordóñez and Stier-Moses [14, 15] defined the robust Wardrop equilibrium for the restrictive case where each user's cost function can be expressed as the sum of two terms: (1) the term depending on the flow but not involving any uncertainty and (2) the term not depending on the flow but involving some uncertainty. They showed that, when the uncertainty set in each route cost functions is polyhedral, the robust TAP can be formulated as an NCP. On the other hand, Takahashi [19] defined the robust Wardrop equilibrium for more general route cost functions without Ordóñez and Stier-Moses' restriction. Moreover, he showed that the robust TAP can be reformulated as a Second-Order Cone Complementarity Problem (SOCCP) [8, 10, 12, 16], when the route cost function is additive, the link cost function is linear and separable[*2] , and the uncertain set is ellipsoidal. Also Takahashi showed that the robust TAP can be reformulated as an MCP when the uncertainty set is defined by means of the ∞-norm.

For the traffic model with uncertain cost functions, Zhang, Chen and Sumalee [21] studied another mathematical approach called a stochastic TAP. They assumed that the uncertain data in the cost functions follow some stochastic distribution, and reformulated the stochastic TAP as a stochastic complementarity problem that can be solved by using the expected residual minimization method. Although Zhang et al. discuss the robustness of the obtained stochastic TAP solution, the meaning of "robust" is essentially different from that in the "robust" TAP model. The robustness in Zhang et al.'s study means that the obtained stochastic TAP solution does not vary so much if the actual value of the stochastic data varies in some degree. On the other hand, the robustness for the robust TAP comes from the "robust optimization", by which each driver chooses his/her route.

In this paper, we consider the robust Wardrop equilibrium in [14, 15, 19] to TAPs with more general uncertainty structures. In [19], Takahashi only considered the case where the link cost functions in traffic network are linear and separable, whereas we study the robust TAP without such a linearity and separability assumption. We also provide the condition for the existence of a robust Wardrop equilibrium, and reformulate the robust TAP as an SOCCP when the uncertainty set is ellipsoidal.

This paper is organized as follows. In Section 2.1, we describe the traffic model and Wardrop's user equilibrium without uncertainty, and formulate the TAP based on the traffic model and Wardrop's user equilibrium. In Section 2.2, we recall background of some equilibrium problems such as SOCCP, MCP, and NCP. In Section 2.3, we formulate the TAP as an NCP and an MCP. Moreover we provide the condition for the existences of a solution for TAP. Section 3 is the main section of this paper. In Section 3.1, we define the robust Wardrop equilibrium, and formulate the robust TAP as an MCP. Furthermore we show the condition for the existence of a solution of the robust TAP. In Section 3.2, we formulate the robust TAP with an ellipsoidal uncertainty set as an SOCCP. In Section 4, we observe the property of equilibria for robust TAPs by means of numerical experiments. In Section 5, we conclude this paper with some remarks.

Throughout the paper, we use the following notations and definitions: $\|\cdot\|$ denotes the 2-norm defined by $\|z\| := \sqrt{z^\top z}$ for a vector z. For a given set S, $|S|$ denotes the cardinality of S. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. For a finite set N and $z = (z_1, z_2, \ldots, z_{|N|})$, we write $z = [z_i]_{i\in N}$. We often write $z = (x, y)$ for $[x^\top, y^\top]^\top$. For the vectors a and b of the same dimension, $a \perp b$ means $a^\top b = 0$.

2 Preliminaries

In this section, we recall some fundamental background on the TAP and some related topics. In Subsection 2.1, we give a mathematical expression of the TAP by using Wardrop's user equilibrium principle. In Subsection 2.2, we introduce some classes of complementarity problems, which play an important role in solving TAPs and robust TAPs. In Subsection 2.3, we reformulate the TAP as a complementarity problem, and study the condition under which TAP solutions exist.

[*2]The link cost function is said to be separable if its value depends only on the link flow.

2.1 Mathematical formulation of traffic assignment problem

In this section, we provide a mathematical formulation of TAP. Consider a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ corresponding to the traffic network, where $\mathcal{N}$ and $\mathcal{L}$ denote the node (vertex or point) set and the link (edge or arc) set, respectively. In the real traffic network, the nodes correspond to the origins, the destinations and the intersections, and the links correspond to the roads. W denotes the set which consists of origin-destination pairs (OD pairs). We assume that graph $\mathcal{G}$ is strongly connected, that is, there exists at least one route for every OD pair $w \in W$. Let R_w be the set of all routes between OD pair $w \in W$, and $R := \cup_{w \in W} R_w$. For $r \in R$, $\mathcal{L}_r \subset \mathcal{L}$ denotes the set of all links contained in r. $y_l \in \mathbb{R}$ and $x_r \in \mathbb{R}$ denote the flow of link $l \in \mathcal{L}$ and route $r \in R$, respectively. Let the link and the route flow vectors be denoted as $y := (y_1, y_2, \dots, y_{|\mathcal{L}|})$ and $x := (x_1, x_2, \dots, x_{|R|})$, respectively. $f_r : \mathbb{R}^{|R|} \to \mathbb{R}$ denotes the cost function for route $r \in R$ with variable $x \in \mathbb{R}^{|R|}$. $t_l : \mathbb{R}^{|\mathcal{L}|} \to \mathbb{R}$ denotes the cost function for link $l \in \mathcal{L}$ with variable $y \in R^{|\mathcal{L}|}$. For an OD pair $w \in W$, $\lambda_w := \min_{r \in R_w} f_r(x)$ denotes the minimum route cost. $d_w : \mathbb{R}^{|W|} \to \mathbb{R}^{|W|}$ denotes the demand function with variable $\lambda := [\lambda_w]_{w \in W}$.

Next, we describe Wardrop's user equilibrium principle which shows drivers' behavior in the traffic network. A route flow vector $x \in \mathbb{R}^{|R|}$ is called Wardrop's user equilibrium if it satisfies

$$[x_r > 0 \implies f_r(x) \leq f_{r'}(x) \quad \forall r' \in R_w] \quad r \in R_w, \ w \in W. \tag{2.1}$$

Wardrop's user equilibrium principle states that each driver in the network selects the route with minimum cost. Conversely, the drivers avoid the routes with non-minimum cost. In other words, under such an equilibrium, the cost of the route with non-zero flow must be less than or equal to other routes for the same OD pair, and conversely, any route with non-minimum cost for an OD pair has no flow.

In addition to Wardrop's user equilibrium principle (2.1), the TAP requires the condition that every route flow is nonnegative and the sum of route flows for each OD pair w is equal to its traffic demand $d_w(\lambda)$, that is,

$$x \geq 0, \quad \sum_{r \in R_w} x_r = d_w(\lambda) \quad (w \in W). \tag{2.2}$$

Combining (2.1) with (2.2), the TAP can be formulated as follows:

$$\begin{aligned} \text{Find} \quad & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq f_r(x) - \lambda_w \perp x_r \geq 0 \quad (r \in R_w, \ w \in W), \\ & \sum_{r \in R_w} x_r = d_w(\lambda) \quad (w \in W), \\ & \lambda_w \geq 0 \quad (w \in W). \end{aligned} \tag{2.3}$$

$$\begin{aligned} \text{Find} \quad & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq f_r(x) - \lambda_w \perp x_r \geq 0 \quad (r \in R_w, \ w \in W), \\ & \sum_{r \in R_w} x_r = d_w(\lambda) \quad (w \in W), \\ & \lambda_w \geq 0 \quad (w \in W). \end{aligned} \tag{2.4}$$

Furthermore, TAP (2.4) can be rewritten equivalently as follows:

$$\begin{aligned} \text{Find} \quad & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq f(x) - N^\top \lambda \perp x \geq 0, \\ & Nx - d(\lambda) = 0, \quad \lambda \geq 0, \end{aligned} \tag{2.5}$$

where function $f : \mathbb{R}^{|R|} \to \mathbb{R}^{|R|}$ and matrix $N \in \mathbb{R}^{|W| \times |R|}$ are defined by

$$f(x) := [f_r(x)]_{r \in R}, \quad N_{wr} = \begin{cases} 1 & r \in R_w \\ 0 & r \notin R_w \end{cases}, \tag{2.6}$$

respectively.

2.2 Complementarity problems

In this subsection, we introduce some classes of complementarity problems [9]. The complementarity problem is a kind of equilibrium problem, and has been studied extensively so far since it is mathematically tractable and can be solved efficiently by existing algorithms such as the smoothing Newton method. In the subsequent sections, we reformulate robust TAPs as complementarity problems.

For given functions $h : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n \times \mathbb{R}^\nu$, NCP and MCP can be formulated as

$$\begin{array}{rl} \text{Find} & x \in \mathbb{R}^n \\ \text{such that} & 0 \le x \perp h(x) \ge 0, \end{array} \tag{2.7}$$

and

$$\begin{array}{rl} \text{Find} & (x, y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \\ \text{such that} & 0 \le x \perp y \ge 0,\ F(x, y, \zeta) = 0, \end{array} \tag{2.8}$$

respectively. Notice that MCP contains NCP as a subclass since NCP (2.7) reduces to MCP (2.8) by setting $F(x, y, \zeta) := y - h(x)$.

The second-order cone complementarity problem (SOCCP) [8, 10, 12, 16] is a more general class of complementarity problems written as follows:

$$\begin{array}{rl} \text{Find} & (x, y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \\ \text{such that} & \mathcal{K} \ni x \perp y \in \mathcal{K}, \quad F(x, y, \zeta) = 0, \end{array} \tag{2.9}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n \times \mathbb{R}^\nu$ is a given function, and $\mathcal{K}$ is the Cartesian product of several second-order cones, that is, $\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$, and the n_i-dimensional second-order cone $\mathcal{K}^{n_i} \subset \mathbb{R}^{n_i}$ is defined as

$$\mathcal{K}^{n_i} = \left\{ (z_1, z_2^\top)^\top \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid z_1 \ge \|z_2\| \right\}.$$

Notice that SOCCP contains MCP as a subclass since $\mathcal{K}$ coincides with the nonnegative orthant when $n_1 = n_2 = \cdots = n_m = 1$. In this paper, we formulate the robust TAP as an SOCCP of the form

$$\begin{array}{rl} \text{Find} & \zeta \in \mathbb{R}^\nu \\ \text{such that} & \mathcal{K} \ni G(\zeta) \perp H(\zeta) \in \mathcal{K},\ C\zeta = h, \end{array} \tag{2.10}$$

where $G : \mathbb{R}^\nu \to \mathbb{R}^n$ and $H : \mathbb{R}^\nu \to \mathbb{R}^n$ are given functions, and $C \in \mathbb{R}^{\nu \times \nu}$ and $h \in \mathbb{R}^\nu$ are given constants. We can easily see that SOCCP (2.10) can be rewritten as SOCCP (2.9) by letting $x := G(\zeta)$, $y := H(\zeta)$, and

$$F(x, y, \zeta) := \begin{bmatrix} x - G(\zeta) \\ y - H(\zeta) \\ C\zeta - d \end{bmatrix}.$$

2.3 Complementarity reformulation of TAP and existence of solution

In this section, we show some relation between TAP (2.5) and NCP (2.7) or MCP (2.8), and discuss the existence of a TAP solution. In order to formulate the TAP as an NCP,we make the following assumption.

Assumption A *In TAP (2.5), the following conditions hold:*

(a) $f(x) \ge 0$ *and* $d(\lambda) \ge 0$ *for any* $(x, \lambda) \in \mathbb{R}_+^{|R|} \times \mathbb{R}_+^{|W|}$,

(b) *For all* $r \in R$, $f_r(x)x_r = 0$ *implies* $x_r = 0$.

Notice that (b) automatically holds if $f(x) > 0$ for any $x \in \mathbb{R}_+^{|R|}$. Under this assumption, TAP (2.5) can be rewritten in the form of NCP (2.7).

Theorem 2.1 [9, Proposition 1.4.6] *Suppose that TAP (2.5) satisfies Assumption* A. *Then, the TAP can be reformulated as the following NCP equivalently:*

$$\begin{aligned} \text{Find} \quad & (x,\lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq \begin{bmatrix} f(x) - N^\top \lambda \\ Nx - d(\lambda) \end{bmatrix} \perp \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq 0. \end{aligned} \tag{2.11}$$

By using the above NCP reformulation, we can derive a sufficient condition under which there exists at least one solution of TAP (2.5).

Assumption B *In TAP (2.5), functions f and d are continuous. Moreover, there exists $M > 0$ such that $d_w(\lambda) \leq M$ for any $w \in W$ and $\lambda \in \mathbb{R}^{|W|}$.*

Theorem 2.2 [9, Proposition 2.2.14] *Suppose that Assumptions* 2.1 *and* B *hold. Then, TAP (2.5) has at least one solution.*

We have shown that TAP (2.5) reduces to an NCP under Assumption 2.1. On the other hand, it also reduced to an MCP under another assumption. In the subsequent numerical experiments, in order to some (robust) TAPs, we apply a smoothing Newton algorithm to this MCP.

Assumption C *In TAP (2.5), It follows $f(x) \geq 0$ and $d(\lambda) > 0$ for any $(x,\lambda) \in \mathbb{R}_+^{|R|} \times \mathbb{R}_+^{|W|}$.*

Theorem 2.3 *Suppose that TAP (2.5) satisfies Assumption* C. *Then, the TAP can be reformulated as the following MCP equivalently:*

$$\begin{aligned} \text{Find} \quad & (x,\lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq f(x) - N^\top \lambda \perp x \geq 0, \\ & Nx = d(\lambda). \end{aligned} \tag{2.12}$$

3 Traffic assignment problem based on robust Wardrop's user equilibrium

In this section, we define the robust TAP and discuss the existence of its solution. We also formulate the robust TAP with special uncertainty structure as an SOCCP.

3.1 Robust traffic assignment problem and existence of solutions

In this subsection, we provide a mathematical expression of the robust TAP, and study the existence of a robust Wardrop equilibrium.

Consider the following situation. The cost function $f_r^{\hat{u}^r}$ for route $r \in R$ contains uncertain data $\hat{u}^r$. Even though the users cannot estimate the value of $\hat{u}^r$ accurately, they know that it belongs to a certain compact set U_r. In such a situation, we assume that each user with OD pair w chooses a route with minimum worst cost, i.e., a route r such that $r = \operatorname{argmin}_{r \in R_w} \tilde{f}_r(x)$, where

$$\tilde{f}_r(x) := \max \left\{ f_r^{\hat{u}^r}(x) \mid \hat{u}^r \in U_r \right\} \tag{3.1}$$

is called the *worst cost function.* Moreover, a Wardrop equilibrium with respect to the worst cost $\tilde{f}_r(x)$ is called a *robust Wardrop equlibrium.*

Definition 3.1 *Let the worst cost function $\tilde{f}_r$ be defined by* (3.1)*. Then, a route flow vector $x \in \mathbb{R}^{|R|}$ satisfying*

$$[x_r > 0 \Longrightarrow \tilde{f}_r(x) \leq \tilde{f}_{r'}(x) \quad \forall r' \in R_w] \quad (r \in R_w,\ w \in W), \tag{3.2}$$

is called a robust Wardrop equlibrium. Moreover, the problem of finding the robust Wardrop equilibrium is called a robust TAP, i.e., it is to find $(x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|}$ such that

$$\begin{aligned} 0 \leq \tilde{f}_r(x) - \lambda_w \perp x_r \geq 0 \quad (r \in R_w,\ w \in W), \\ \sum_{r \in R} x_r = d_w(\lambda),\ \lambda_w \geq 0 \quad (w \in W). \end{aligned} \tag{3.3}$$

By using the complementarity reformulation technique in the previous section, we can also show conditions under which a robust Wardrop equilibrium exists. In what follows, we denote $\tilde{f}(x) := [\tilde{f}_r(x)]_{r=1}^{|R|} \in \mathbb{R}_+^{|R|}$.

Assumption D *For the robust TAP*(3.3)*, the following four conditions hold:*

(a) *For any $x \in \mathbb{R}_+^{|R|}$, there exists $\hat{u}^r \in U_r$ such that $f_r^{\hat{u}^r}(x) > 0$ for each $r \in R$.*

(b) *For each $r \in R$, the function $h_r : \mathbb{R}_+^{|R|} \times U_r \to \mathbb{R}_+$ defined by $h_r(x, \hat{u}^r) := f_r^{\hat{u}^r}(x)$ is continuous on $\mathbb{R}_+^{|R|} \times U_r$.*

(c) *$d(\lambda) > 0$ for any $\lambda \in \mathbb{R}_+^{|W|}$.*

(d) *For each $w \in W$, function $d_w(\lambda)$ is continuous and bounded above on $\mathbb{R}^{|W|}$.*

Theorem 3.1 *Suppose that Assumption* D *holds. Then the robust TAP*(3.3) *is equivalent to the following MCP:*

$$\begin{aligned} \text{Find} \quad & (x, \lambda) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \\ \text{such that} \quad & 0 \leq \tilde{f}(x) - N^\top \lambda \perp x \geq 0, \\ & Nx - d(\lambda) = 0. \end{aligned} \tag{3.4}$$

3.2 SOCCP reformulation for robust TAP with ellipsoidal uncertainty sets

In the previous subsection, we have defined the robust TAP and showed the condition for the existence of a solution. In this section, we show that the robust TAP can be reformulated as an SOCCP when the uncertainty sets are described by means of the Euclidean norm.

3.2.1 Robust TAP with general link cost function

In what follows, we assume that each link cost function is expressed as

$$t_l^{\hat{u}_l}(y) = t_l(y) + \hat{u}_l \Delta t_l(y), \tag{3.5}$$

where $t_l : \mathbb{R}^{|\mathcal{L}|} \to \mathbb{R}$ and $\Delta t_l : \mathbb{R}^{|\mathcal{L}|} \to \mathbb{R}$ are given functions, and $\hat{u}_l \in \mathbb{R}$ denotes the uncertainty parameter. Moreover, we suppose that the uncertain route cost function $f_r^{\hat{u}^r}(x)$ is additive, i.e.,

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathcal{L}_r} t_l^{\hat{u}_l}(y), \tag{3.6}$$

where the uncertainty parameter satisfies $\hat{u}^r = [\hat{u}_l]_{l \in \mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|}$. Now, let $M \in \mathbb{R}^{|\mathcal{L}| \times |R|}$ be the link-route incidence matrix with the (l, r) entry

$$M_{lr} := \begin{cases} 1 & (l \in \mathcal{L}_r) \\ 0 & (l \notin \mathcal{L}_r). \end{cases}$$

Then we have $y = Mx$, which together with (3.5) and (3.6) yields

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathcal{L}_r} t_l(Mx) + \hat{u}_l \Delta t_l(Mx). \tag{3.7}$$

Furthermore, we make the following assumption on the uncertainty set U_r.

Assumption E *Uncertainty set U_r is ellipsoidal for each $r \in R$, i.e.,*

$$U_r := \left\{ \hat{u}^r \in \mathbb{R}^{|\mathcal{L}|} \,\middle|\, \hat{u}^r = \bar{u}^r + D_r \hat{v}^r, \ \|\hat{v}^r\| \leq \delta_r, \right\},$$

where $\bar{u}^r$ is a given vector, $D_r \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|}$ is a given symmetric positive definite matrix, and δ_r is a given positive scalar.

Under Assumption E, we can represent the worst cost function $\tilde{f}_r$ explicitly. To this end, we need the following lemma.

Lemma 3.1 *Let $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ be arbitrary vectors, $C \in \mathbb{R}^{m \times n}$ be an arbitrary matrix, and $\delta > 0$ be any positive scalar. Let $P \subset \mathbb{R}^m$ be defined by*

$$P := \left\{ p \in \mathbb{R}^m \,\middle|\, p = b + Cq, \|q\| \leq \delta \right\}.$$

Then we have

$$\max_{p \in \mathbb{R}^m} \left\{ a^\top p \,\middle|\, p \in P \right\} = a^\top b + \delta \|C^\top a\|. \tag{3.8}$$

Applying Lemma 3.1 to the uncertain route cost $f_r^{\hat{u}^r}$ with (3.7) under Assumption E, we readily obtain

$$\tilde{f}_r(x) = \sum_{l \in \mathcal{L}_r} t_l(Mx) + \bar{u}_l^r \Delta t_l(Mx) + \delta_r \|D_r \mathrm{diag}(M_r) \Delta t(Mx)\|, \tag{3.9}$$

where $\mathrm{diag}\,(M_r) \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|}$ is the diagonal matrix whose diagonal components are given by M_{lr} $(l \in \mathcal{L})$.

By Theorem 3.1, the robust TAP with $\tilde{f}_r(x)$ defined by (3.9) reduces to MCP (3.4) under Assumption D. However, since $\tilde{f}$ is nondifferentiable, it is difficult to apply existing algorithms to MCP (3.4) directly. To avoid this difficulty, we reformulate the robust TAP as an SOCCP that contains differentiable functions only.

Let $g_r(x) := \sum_{l \in \mathcal{L}_r} t_l(Mx) + \bar{u}_l^r \Delta t_l(Mx)$ and $g(x) := [g_r(x)]_{r=1}^{|R|} \in \mathbb{R}^{|R|}$. Then the worst cost function (3.9) can be expressed explicitly as

$$\tilde{f}(x) = g(x) + \begin{bmatrix} \delta_1 \|D_1 \mathrm{diag}(M_1) \Delta t(Mx)\| \\ \delta_2 \|D_2 \mathrm{diag}(M_2) \Delta t(Mx)\| \\ \vdots \\ \delta_{|R|} \|D_{|R|} \mathrm{diag}(M_{|R|}) \Delta t(Mx)\| \end{bmatrix}.$$

Moreover, by using an auxiliary variable $s := [s_r]_{r=1}^{|R|} \in \mathbb{R}^{|R|}$, MCP (3.4) can be rewritten as the following problem:

$$\begin{aligned} \text{Find} \quad & (x, \lambda, s) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \times \mathbb{R}^{|R|} \\ \text{such that} \quad & 0 \leq g(x) + s - N^\top \lambda \perp x \geq 0, \\ & s_r = \delta_r \|D_r \mathrm{diag}(M_r) \Delta t(Mx)\| \ \ (r \in R), \qquad (3.10) \\ & Nx = d(\lambda). \end{aligned}$$

(3.11)

Furthermore, we can reformulate (3.10) as an SOCCP by the following lemma.

Lemma 3.2 *Let $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^{k-1}$ be an arbitrary vector with $k \geq 2$. Then, $\xi_1 = \|\xi_2\|$ if and only if there exists a vector $v \in \mathbb{R}^{k-1}$ such that*

$$\mathcal{K}^k \ni \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ v \end{bmatrix} \in \mathcal{K}^k. \tag{3.12}$$

By Lemma 3.2, problem (3.10) can be reformulated as the following SOCCP:

$$\begin{aligned} \text{Find} \quad & (x, \lambda, s, v) \in \mathbb{R}^{|R|} \times \mathbb{R}^{|W|} \times \mathbb{R}^{|R|} \times \mathbb{R}^{|\mathcal{L}||R|} \\ \text{such that} \quad & 0 \leq g(x) + s - N^\top \lambda \perp x \geq 0, \\ & \mathcal{K}^{|\mathcal{L}|+1} \ni \begin{bmatrix} s_r \\ \delta_r D_r \mathrm{diag}(M_r) \Delta t(Mx) \end{bmatrix} \perp \begin{bmatrix} 1 \\ v^r \end{bmatrix} \in \mathcal{K}^{|\mathcal{L}|+1} \quad (r \in R), \\ & Nx = d(\lambda). \end{aligned} \tag{3.13}$$

Moreover, if d is a constant function, i.e., $d(\lambda) = d$ for any $\lambda \in \mathbb{R}^{|W|}$, then SOCCP (3.13) can be rewritten in the form of SOCCP (2.10) with $\mathcal{K} := (\mathcal{K}^1)^{|R|} \times (\mathcal{K}^{|\mathcal{L}|+1})^{|R|}$, $\zeta := (x, \lambda, s, v) \in \mathbb{R}^{|R|+|W|+|R|(|\mathcal{L}|+1)}$,

$$C = \begin{bmatrix} 0_{|R|\times|R|} & 0_{|R|\times(|W|+|R|(|\mathcal{L}|+1))} \\ N & 0_{|R|\times(|W|+|R|(|\mathcal{L}|+1))} \\ 0_{|R|(|\mathcal{L}|+1)\times|R|} & 0_{|R|\times(|W|+|R|(|\mathcal{L}|+1))} \end{bmatrix}, \quad h = \begin{bmatrix} 0_{|R|} \\ d \\ 0_{|R|(|\mathcal{L}|+1)} \end{bmatrix},$$

$$F(x, \lambda, s, v) := \begin{bmatrix} g(x) + s - N^\top \lambda \\ s_1 \\ \delta_1 D_1 \mathrm{diag}(M_1) \Delta t(Mx) \\ s_2 \\ \delta_2 D_2 \mathrm{diag}(M_2) \Delta t(Mx) \\ \vdots \\ s_{|R|} \\ \delta_{|R|} D_{|R|} \mathrm{diag}(M_{|R|}) \Delta t(Mx) \end{bmatrix}, \quad G(x, \lambda, s, v) := \begin{bmatrix} x \\ 1 \\ v^1 \\ 1 \\ v^2 \\ \vdots \\ 1 \\ v^{|R|} \end{bmatrix},$$

where 0_m and $0_{m\times n}$ are the m-dimensional zero vector and the $(m \times n)$-dimensional zero matrix, respectively.

3.2.2 Robust TAP with uncertain BPR function

Next we introduce a more concrete link cost function called the U. S. Bureau of Public Roads (BPR) function [7]. The BPR function $t_l(y)$ is defined as follows:

$$t_l(y) = a_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^\nu\right), \tag{3.14}$$

where ν, a_l, b_l, c_l are positive scalars. More precisely, a_l represents the free-flow travel time, b_l represents the congestion factor, c_l represents the traffic capacity of link l, and ν is usually chosen as a number between 4 and 5. The BPR function is one of the most popular link cost functions employed in a mathematical model for the traffic network. We suppose that for all routes $r \in R$, the cost functions $f_r(x)$ are additive. Then by using the BPR function, we can express the route cost function as follows:

$$f_r(x) = \sum_{l \in \mathcal{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^\nu\right), \tag{3.15}$$

where M_l denotes the l-th row vector of the link-route incidence matrix M.

Now we consider the situation where the data in the BPR function (3.14) involve uncertainties. Then we formulate such a robust TAP as an SOCCP. In the remainder of this section, we suppose that Assumption E holds for the uncertainty set.

Uncertainty in the traffic capacity We consider the situation that the traffic capacity c_l is uncertain. Specifically we suppose that c_l is expressed as $c_l = \bar{c}_l + \hat{u}_l$ with nominal $\bar{c}_l$ and uncertainty parameter $\hat{u}_l \in \mathbb{R}$.

Then, the link cost and route cost functions can be expressed as

$$t_l^{\hat{u}_l}(y) = a_l \left(1 + b_l \left(\frac{M_l x}{\bar{c}_l + \hat{u}_l}\right)^\nu\right), \tag{3.16}$$

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathcal{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l + \hat{u}_l}\right)^\nu\right), \tag{3.17}$$

respectively. Here we assume that $\hat{u}_l > -\bar{c}_l$ so that the denominator will not be zero.

In order to obtain the SOCCP reformulation, we had to assume that the uncertain link cost function is expressed as (3.5). However, function $t_l^{\hat{u}_l}$ in (3.16) cannot be written in the form (3.5) in a straightforward manner. We therefore introduce an "approximate link cost function" based on the first-order Taylor expansion as follows:

$$t_l^{\hat{u}_l}(y) := a_l \left(1 + b_l \left(\frac{M_l x}{\bar{c}_l}\right)^\nu\right) - \frac{\nu a_l b_l (M_l x)^\nu}{\bar{c}_l^{\nu+1}} \hat{u}_l. \tag{3.18}$$

Also the approximate route cost function can be expressed as

$$f_r^{\hat{u}^r}(x) := \sum_{l \in \mathcal{L}_r} a_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^\nu\right) - \sum_{l \in \mathcal{L}_r} \frac{\nu a_l b_l (M_l x)^\nu}{c_l^{\nu+1}} \hat{u}_l. \tag{3.19}$$

Since the uncertainty parameter $\hat{u}_l$ is very small in general, this approximation is reasonable. Now, let

$$\Delta t_l(y) = -\frac{\nu a_l b_l (M_l x)^\nu}{c_l^{\nu+1}}.$$

Then, (3.18) and (3.19) correspond to (3.5) and (3.7), respectively. Thus, we can reformulate the robust TAP as an SOCCP by using the results of Subsection 3.1.2.

Uncertainty in the free-flow travel time We consider the situation that the free-flow travel time a_l is uncertain. Specifically we suppose that a_l is expressed as $a_l = \bar{a}_l + \hat{u}_l$ with nominal value $\bar{a}_l$ and uncertainty parameter $\hat{u}_l \in \mathbb{R}$. Then, the link and route cost functions can be expressed as

$$t_l^{\hat{u}_l}(y) = \bar{a}_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^\nu\right) + \hat{u}_l \left(1 + b_l \left(\frac{y_l}{c_l}\right)^\nu\right), \tag{3.20}$$

$$f_r^{\hat{u}^r}(x) = \sum_{l \in \mathcal{L}_r} \bar{a}_l \left(1 + b_l \left(\frac{M_l x}{c_l}\right)^\nu\right) + \sum_{l \in \mathcal{L}_r} \hat{u}_l \left(1 + \left(\frac{M_l x}{c_l}\right)^\nu\right), \tag{3.21}$$

respectively. Let

$$\Delta t_l(y) := 1 + \left(\frac{M_l x}{c_l}\right)^\nu.$$

Then (3.20) and (3.21) correspond to (3.5) and (3.7), respectively. Thus, we can reformulate the robust TAP as an SOCCP by using the results in Subsection 3.2.1.

4 Numerical experiments

In this section, we introduce two specific traffic models with uncertainty set of various sizes in the cost functions defined by (3.14) and (3.15). We try to compute robust Wardrop equilibria by using the SOCCP reformulation approach studied in Section 3.2 and observe their properties. Throughout this section, we let the uncertainty set U_r $(r \in R_w)$ be given by

$$U_r := \left\{ \hat{u}^r \in \mathbb{R}^{|E|} \,\middle|\, \|\hat{u}^r\| \le \rho_w \right\}, \tag{4.1}$$

where ρ_w is a positive constant for each $w \in W$. Notice that OD pair is identified for each $r \in R$. Also, we consider the case with $\nu = 4$ in the cost functions defined by (3.14) and (3.15).

For solving the SOCCPs, we apply the Newton-type method that uses a smoothing technique [12]. All programs are coded in MATLAB 2010a and run on a machine with Intel® Core i5 430M 2.27GHz CPU and 4.00GB memories.

Relationship between size of uncertainty sets and robust Wardrop equilibria

We consider the traffic model illustrated in Figure 1. Each node denotes an origin, a destination, and an intersection, and each link denotes a road connecting the nodes. The set of OD pairs is given by $W := \{w_1, w_2\}$, where $w_1 = (1 \to 5)$ and $w_2 = (2 \to 6)$. The demands for w_1 and w_2 are given by $d_{w_1} = d_{w_2} = 10$. We suppose that the demands do not depend on λ. We give the routes $r \in R = R_{w_1} \cup R_{w_2}$, and the coefficients a_l, b_l, and c_l of the link functions (3.14) as shown in Table 1 and 2, respectively. Now we consider the case where only a_l is uncertain with uncertainty set U_r expressed by (4.1). Therefore, we use (3.20) and (3.21) as the link cost function and the route cost function with uncertainty, respectively. In this experiment, we vary ρ_{w_1} from 0.001 to 5, and fix ρ_{w_2} at 0.001, and compute a robust Wardrop equilibrium for each ρ_{w_1}. Then, we observe the route flow $\{x_r\}_{r \in R}$ and the minimum cost λ_w at the obtained equilibria of the robust TAPs.

Table 3 shows the obtained values of $\{x_r\}_{r \in R}$ and $\{\lambda_w\}_{w \in W}$ at the equilibrium for each ρ_{w_1}. From the table, we can observe that, as ρ_{w_1} increases, x_{r_1} and λ_{w_1} get larger, but x_{r_2} gets smaller. On the other hand, as to w_2, as ρ_{w_1} increases, x_{r_4} and λ_{w_2} get smaller, but x_{r_3} gets larger. We can interpret these results as follows: Let us consider the drivers who belong to the OD pair $w_1 \in W$. In Figure 1, r_1 has only one link 1, while r_2 has three links 2, 3 and 4, that is, route r_2 is more complicated than r_1. In such a situation, drivers may think that more complicated routes involve more uncertainty and require higher costs than simple routes, and therefore avoid using route r_2. Thus the result of this experiment well reflect such driver's estimation for uncertainty.

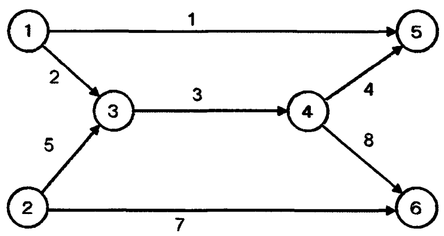

Figure 1: The network in section 4

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Table 1: Relation of OD pairs, routes and links in Figure 1

OD pair	route	order of links
w_1	r_1	1
	r_2	$2 \to 3 \to 4$
w_2	r_3	$5 \to 3 \to 8$
	r_4	7

Table 2: Coefficients of link cost functions

link	a_l	b_l	c_l
1	5	0.15	2
2	1	0.15	1
3	1	0.15	1
4	1	0.15	1
5	1	0.15	1
6	1	0.15	1
7	5	0.15	2

Table 3: Uncertainty size, obtained route flow and minimum cost ($\rho_{w_2} = 0.001$)

ρ_{w_1}	x_{r_1}	x_{r_2}	x_{r_3}	x_{r_4}	λ_{w_1}	λ_{w_2}
0	7.329	2.671	2.670	7.330	140.268	140.369
0.001	7.330	2.670	2.670	7.330	140.345	140.345
0.005	7.333	2.667	2.671	7.329	140.654	140.250
0.01	7.336	2.664	2.673	7.327	141.039	140.131
0.05	7.362	2.638	2.685	7.315	144.114	139.206
0.1	7.393	2.607	2.701	7.299	147.938	138.104
0.5	7.603	2.397	2.800	7.200	177.753	131.003
1	7.793	2.207	2.887	7.113	213.425	125.000
5	8.378	1.622	3.137	6.863	471.863	109.038

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